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Infrared Asymptotic Freedom of a Hierarchical ϕ_3^6 Lattice Theory

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For the weakly coupled lattice ϕ_3^6 theory in a hierarchical model approximation a nonperturbative renormalization group analysis in the spirit of Gawędzki and Kupiainen is performed to study the flow of the effective actions. We deduce a domain of attraction to the tricritical (Gaussian) fixed point. The two relevant coupling constants of the problem are controlled by analytic continuation to complex domains, tracing their images under the renormalization group iterations.

KEY WORDS: Tricritical ϕ_3^6 lattice theory; hierarchical model approximation; renormalization group; block spins.

1. INTRODUCTION AND RESULTS

The concepts of the renormalization group have become the major theoretical perspective in analyzing the long range behavior of systems with an infinite number of degrees of freedom. However, the instrumentation of these concepts for particular models constitutes a formidable task. In order to simplify the problem for scalar field theories Wilson introduced an approximate renormalization group recursion relation[1,2] which in a formal perturbation expansion still produces essentially the same set of graphs as the full problem. Using this framework Riedel and Wegner[3] have already treated a $d=3$ scalar lattice theory with a potential formed by an even sixth order polynomial as a model for tricritical behavior. They found on a perturbative level the flow to a massless Gaussian fixed point.

On the theoretical side a very enlightening approach to the renormalization group has been developed by Gawędzki and Kupiainen.[4] They decomposed the scalar massless lattice field into massive fluctuations on

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successive block lattices which are nearly identically distributed in their respective scales. Guided by these properties these authors introduce a set of models replacing the lattice Laplacian by an approximate version of it which implies a hierarchical structure. It is the merit of such hierarchical models that a local action is strictly transformed by Wilson–Kadanoff block spin transformations into again a local one. Denoting by $g^{(n)}(\phi)$ the Gibbs factor connected with the effective potential at a lattice site and by $L=2, 4, 6,\ldots$ the necessarily even block length, the renormalization group of these hierarchical models for a scalar field $\phi \in \mathbb{R}$ in dimension d reads, $n \in \mathbb{N}_0$,

$$g^{(n+1)}(\phi)=\frac{\int_{-\infty}^{\infty} d\mu(z)[g^{(n)}(L^{-1/2(d-2)}\phi+z)\, g^{(n)}(L^{-1/2(d-2)}\phi-z)]^{L^d/2}}{\int_{-\infty}^{\infty} d\mu(z)\, g^{(n)}(z)^{L^d}} \tag{1.1}$$

with the Gaussian measure (on $\mathbb{R}$)

$$d\mu(z)=\frac{1}{\sqrt{2\pi}}\, e^{-(1/2)z^2}\, dz \tag{1.2}$$

It is remarkable that writing $g^{(n)}(\phi)=\exp[-L^{-d}Q_n(2^{-1/2}\phi)]$ transforms (1.1) in the case of $L=2$ into Wilson's approximate recursion relation. Moreover Gawędzki and Kupiainen invented a powerful analyticity technique,[5] devised to control nonperturbatively the flow of the renormalization group (1.1), necessary in a rigorous treatment because of the unbounded scalar field. Employing this method they showed, among other results, the infrared asymptotic freedom of the weakly coupled $d=4$ lattice ϕ^4 theory within the hierarchical model approximation (1.1).[5,6] Very recently these authors could extend their analysis beyond the hierarchical model approximation treating the full massless ϕ_4^4 theory.[7]

The subject of our work is to investigate rigorously a weakly coupled ϕ^6 theory in $d=3$ within the hierarchical renormalization group (1.1), starting with a family of Gibbs factors

$$g^{(0)}(\phi)=e^{-(\rho_0/2)\phi^2-(\lambda_0/6)\phi^4-(\sigma_0/15)\phi^6} \tag{1.3}$$

where ρ_0, λ_0 are complex and σ_0 real positive and sufficiently small. We perform in close analogy to Gawędzki and Kupiainen[5] analytic continuations of the Gibbs factors $g^{(n)}(\phi)$ to complex values of the field ϕ, but in addition control their holomorphic dependence on the initial coupling constants $(\rho_0, \lambda_0)\in \mathscr{D}_0 \subset \mathbb{C}^2$, where $\mathscr{D}_0$ is a suitably chosen compact set. Actually a slightly larger family of initial Gibbs factors sufficiently "close" to (1.3) is considered, stated in Section 2. Exploiting the fact that the Gibbs factors $g^{(n)}(\phi)$ are holomorphic functions of (ρ_0, λ_0) we show existence and uniqueness of (real) critical values $(\rho_0, \lambda_0)=(\rho^*, \lambda^*)$ for which convergence to the infrared asymptotically free fixed point $g \equiv 1$ occurs. Following the iteration, we will learn how to restrict (ρ_0, λ_0) successively to compact sets $\mathscr{D}_n \subset \mathscr{D}_{n-1} \subset \cdots \subset \mathscr{D}_0$, shrinking exponentially in diameter, in order to have bounds on the expansive parameters (ρ_n, λ_n) in $g^{(n)}(\phi)$. The critical point is found as

$$\{(\rho^*, \lambda^*)\} = \bigcap_{n=0}^{\infty} \mathscr{D}_n \tag{1.4}$$

We formulate our main result in the following theorem.

Theorem. For initial Gibbs factors (1.3) with small $\sigma_0 > 0$ there exist real critical values $\rho_0 = \rho^*(\sigma_0)$, $\lambda_0 = \lambda^*(\sigma_0)$ such that

$$\lim_{\substack{n \uparrow \infty \\ n > \mathscr{N}}} g^{(n)}(\phi) = 1$$

uniformly on any compact domain in $\mathbb{C}$, where $\mathscr{N}$ depends on the domain.

Uniqueness of $\rho^*(\sigma_0)$, $\lambda^*(\sigma_0)$ follows by our method requiring that the expansive parameters remain close to their perturbative values at each iterative step.

Several comments should be added:

(i) Besides the couplings already appearing in (1.3) a ϕ^8 term the coupling of which becomes negative in the course of the iteration has to be traced explicitly. This is necessary because of internal consistency of our estimates; compare (5.20): only the effect of this coupling renders finally the ϕ^6 coupling marginally stable. Nevertheless it is possible to begin with (1.3).

(ii) We find it satisfactory, that our estimates allow all possible values of the block length L, i.e., $L \geqslant 2$, even.

(iii) We emphasize that the simple argument,[8,5] based on continuity, to derive the existence of a critical point in the case of one relevant coupling does not work for more than one. Our analytic technique can obviously be extended to a finite number of relevant couplings.

The article is organized as follows. In Section 2 we state the inductive assumptions for the Gibbs factors. In Section 3 the inductive reproduction of their general analytic properties is shown. The assumptions on large fields are inductively reproduced in Section 4. Finally, in Section 5, forming the nucleus of the approach, we reproduce inductively the assumed small field properties; in Section 5.1 the holomorphic properties of the effective potential are derived and in Section 5.2 the behavior of the coupling constants is controlled.

2. INDUCTIVE ASSUMPTIONS

In order to avoid cumbersome notation we introduce for a general $n \in \mathbb{N}_0$

$$g(\phi) \equiv g^{(n)}(\phi), \qquad g'(\phi) \equiv g^{(n+1)}(\phi) \tag{2.1}$$

for the Gibbs factors and

$$B_n = (n + n_0)^\alpha \tag{2.2}$$

with fixed n_0, $\alpha \in \mathbb{R}_+$. As our succeeding estimates will show, n_0 has to be chosen sufficiently large and $\alpha < 1/12$.

Let $\phi = \phi_1 + i\phi_2 \in \mathbb{C}$ with $\phi_1, \phi_2 \in \mathbb{R}$, and the relevant coupling constants ρ_0 and λ_0 of the initial action, (1.3), chosen complex, $(\rho_0, \lambda_0) \in \mathscr{D}_0 \subset \mathbb{C}^2$. $\mathscr{D}_0$ is a compact simply connected domain restricted in each iteration step to compact domains $\mathscr{D}_n \subset \operatorname{int} \mathscr{D}_{n-1} \subset \mathscr{D}_0$, as will be shown in Section 5.2.

The Gibbs factor $g(\phi)$ is assumed to have the following properties:

(A_1) General Analytic Properties. $g(\phi)$ is a holomorphic function of $(\phi, \rho_0, \lambda_0)$ in the domain $\{\phi \in \mathbb{C}: \ |\phi_2| \leqslant B_n\} \times \mathscr{D}_n \subset \mathbb{C}^3$. Moreover, for fixed ρ_0, λ_0

$$g(-\phi) = g(\phi); \qquad g(0) = 1$$

(A_2) Large Field. For $|\phi| > B_n$ and $|\phi_2| \leqslant \kappa/\sqrt{L}\ B_n$ with $1 < \kappa < \sqrt{4/3}$, such that $\kappa B_n > B_{n+1}$ due to n_0 large enough, the function $g(\phi)$ is bounded uniformly in $(\rho_0, \lambda_0) \in \mathscr{D}_n$ by

$$|g(\phi)| < e^{-(1/15)\operatorname{Re}\hat{\sigma}\{(1/2)\phi_1^6 - \phi_2^2(15\phi_1^4 + \phi_2^4) + D\phi_1^4\}}$$

with $\operatorname{Re}\hat{\sigma} \in \mathbb{R}_+$ specified in the following assumption and D a large fixed positive constant; see (4.21). The restriction on κ stated above implies $|\phi_1| \geqslant B_n/\sqrt{3}$ for all admitted values of L: $L \geqslant 2$, even.

(A_3) Small Field. For $|\phi| \leqslant B_n$ there exists a function $v(\phi)$ holomorphic in $(\phi, \rho_0, \lambda_0)$ such that

$$g(\phi) = e^{-v(\phi)}$$

$$v(\phi) = \frac{\rho}{2}\phi^2 + \frac{\lambda}{6}\phi^4 + \frac{\sigma}{15}\phi^6 + \frac{\tau}{28}\phi^8 + \tilde{v}(\phi)$$

$$\tilde{v}(\phi) = \sum_{l=5}^{\infty} \frac{\phi^{2l}}{(2l)!}\left(\frac{d}{d\phi}\right)^{2l} v(0)$$

The complex coupling constants ρ, λ, σ, τ depending on the complex initial values ρ_0, λ_0 (and the real σ_0) are restricted such that

$$|(n+n_0)\rho|,\ |(n+n_0)\lambda|,\ |(n+n_0)\sigma|,\ \text{and}\ |(n+n_0)^2\tau|$$

have upper bounds uniformly in ρ_0, λ_0, independent of n_0+n, and $(n+n_0)\,\mathrm{Re}\,\sigma > \mathrm{const} > 0$, and $128(3L)^3\,|\mathrm{Im}\,\hat{\sigma}| < \mathrm{Re}\,\hat{\sigma}$. The precise relation between $\hat{\sigma}$ and σ will be stated later in (5.38). It implies that $\mathrm{Re}\,\hat{\sigma}$, which is positive and monotonously decreasing with n, satisfies $\mathrm{Re}\,\hat{\sigma} = \mathrm{Re}\,\sigma + O((n+n_0)^{-2})$. Furthermore $\tilde{v}(\phi)$ is bounded by $|\tilde{v}(\phi)| < a(n+n_0)^{-2}$ uniformly in ϕ, ρ_0, λ_0, with fixed $a \in \mathbb{R}_+$.

3. INDUCTIVE REPRODUCTION OF THE GENERAL ANALYTIC PROPERTIES

We follow the notation of Ref. (5) and write (1.1) for $d=3$ in the form

$$g'(\phi) = \frac{\int d\mu(z)\, f(\phi, z)}{\int d\mu(z)\, f(0, z)} \tag{3.1}$$

with

$$f(\phi, z) = \left[g\left(\frac{\phi}{\sqrt{L}} + z\right) g\left(\frac{\phi}{\sqrt{L}} - z\right)\right]^{L^3/2} \tag{3.2}$$

In order to reproduce inductively the assumptions (A_1) for $g'(\phi)$ we first observe that in the strip $\phi \in \mathbb{C}$, $|\mathrm{Im}\,\phi| \leqslant B_{n+1}$ we need, because of (3.1) and (3.2), the function g at

$$\left|\mathrm{Im}\left(\frac{\phi}{\sqrt{L}} \pm z\right)\right| = \frac{|\phi_2|}{\sqrt{L}} \leqslant \frac{B_{n+1}}{\sqrt{L}} < \frac{\kappa}{\sqrt{L}} B_n \tag{3.3}$$

From the assumptions (A_1)–(A_3) we deduce that for $\phi \in \mathbb{C}$ with $|\mathrm{Im}\,\phi| < \kappa B_n$, $(\rho_0, \lambda_0) \in \mathrm{int}\,\mathscr{D}_n$, and $z \in \mathbb{R}$ the function $f(\phi, z)$ is (i) simultaneously continuous in ϕ, ρ_0, λ_0, z; (ii) holomorphic in $(\phi, \rho_0, \lambda_0)$ for fixed z; (iii) uniformly in ϕ, ρ_0, λ_0, and z bounded by a constant. Hence, due to the finite absolutely continuous measure (1.2) the integral

$$\int d\mu(z)\, f(\phi, z) \tag{3.4}$$

defines holomorphic functions of the variables ϕ, ρ_0, and λ_0 separately and hence a function holomorphic in $(\phi, \rho_0, \lambda_0) \in \{|\phi_2| < \kappa B_n\} \times \mathrm{int}\,\mathscr{D}_n \supset$

$\{|\phi_2| \leqslant B_{n+1}\} \times \mathscr{D}_{n+1}$. It remains to be shown that at $\phi = 0$ this function has no zeros in (ρ_0, λ_0).

We decompose

$$\int d\mu(z)\, f(0, z) = \int_{|z| < B_n} d\mu(z)\, g(z)^{L^3} + \int_{|z| > B_n} d\mu(z)\, g(z)^{L^3} =: I_1 + I_2 \quad (3.5)$$

From (A_2) we obtain, estimating generously

$$|I_2| < \int_{|z| > B_n} d\mu(z) < \sqrt{2}\, e^{-(1/4) B_n^2} \quad (3.6)$$

Due to (A_3) we have

$$I_1 = \int_{|z| < B_n} d\mu(z)\, e^{-L^3 \operatorname{Re} v(z) - iL^3 \operatorname{Im} v(z)} \quad (3.7)$$

Using, valid for $|\psi| < \frac{1}{2}$

$$|e^{\psi} - 1 - \psi| \leqslant |\psi|^2 \quad (3.8)$$

we obtain, again because of (A_3), with $N = n + n_0$

$$I_1 = \int_{|z| < B_n} d\mu(z)\, e^{-L^3 \operatorname{Re} v(z)} \{1 - iL^3 \operatorname{Im} v(z)\} + O(N^{-2}) \quad (3.9)$$

implying an imaginary part of order N^{-1} and real part of order 1, hence

$$|I_1| = \int_{|z| < B_n} d\mu(z)\, e^{-L^3 \operatorname{Re} v(z)} + O(N^{-2}) \quad (3.10)$$

From (3.6) and (3.10) we thus obtain

$$\left| \int d\mu(z)\, f(0, z) \right| > \int_{|z| < B_n} d\mu(z)\, e^{-L^3 \operatorname{Re} v(z)} - |O(N^{-2})|$$

$$> \int_{|z| < B_n} d\mu(z)\, e^{-L^3 p (\operatorname{Re} \sigma / 15) B_n^4 z^2} - |O(N^{-2})| \quad (3.11)$$

with a positive constant p close to 1. This follows again from (A_3).

Furthermore, one derives

$$\int_{|z| < B_n} d\mu(z)\, e^{-L^3 p (\operatorname{Re} \sigma / 15) B_n^4 z^2} > \left[1 + 2(1 + \varepsilon')\, L^3 p \frac{\operatorname{Re} \sigma}{15} B_n^4 \right]^{-1/2} \quad (3.12)$$

for n_0 large enough, with some small positive ε' not depending on n. Hence, collecting (3.11) and (3.12) we arrive at the lower bound

$$\left|\int d\mu(z)\, f(0, z)\right| > \left[1 + 2(1+\varepsilon)\, L^3 p \frac{\operatorname{Re}\sigma}{15} B_n^4\right]^{-1/2} \tag{3.13}$$

valid for n_0 large enough, with another small positive constant $\varepsilon > \varepsilon'$. This bound shows that the denominator in (3.1) is different from zero. Together with the analyticity properties of the function (3.4) we thus have deduced the properties (A_1) with n replaced by $n+1$.

4. INDUCTIVE REPRODUCTION OF THE LARGE FIELD ASSUMPTIONS

In this section we reproduce for $g'(\phi)$ the properties (A_2) indexed by $n+1$. We obtain an upper bound for (3.1) by deriving an upper bound for the numerator and employing the lower bound (3.13) for the denominator.

Because of symmetry we first observe that it suffices in (3.1) to consider $\operatorname{Re}\phi \geqslant 0$ and to restrict the z integration to $0 \leqslant z < \infty$, which will always be done in this section. Introducing the shorthand notations

$$x = \frac{\phi_1}{\sqrt{L}}, \qquad y = \frac{\phi_2}{\sqrt{L}}, \qquad \tilde{\sigma} = \frac{1}{15} \operatorname{Re}\hat{\sigma}, \qquad \tilde{\tilde{\sigma}} = \frac{1}{15} \operatorname{Im}\hat{\sigma} \tag{4.1}$$

we obtain, according to the assumptions (A_2) and (A_3) respectively, separate estimates in the case of

$$|x + iy \pm z| > B_n:$$

$$\left| g\left(\frac{\phi}{\sqrt{L}} \pm z\right)\right| < e^{-\tilde{\sigma}\{1/2(x\pm z)^6 - y^2[15(x\pm z)^4 + y^4] + D(x\pm z)^4\}} \tag{4.2}$$

called "large case," and in the case of

$$|x + iy \pm z| \leqslant B_n:$$

$$\left| g\left(\frac{\phi}{\sqrt{L}} \pm z\right)\right| < e^{\tilde{\sigma}\{O(B_n^4) - \operatorname{Re}(x+iy\pm z)^6\} + |\tilde{\tilde{\sigma}}| B_n^6} \tag{4.3}$$

called "small case." In (4.3) the order $O(B_n^4)$ estimates the remaining terms occurring in (A_3) as well as the effect of replacing σ by $\hat{\sigma}$. Moreover we used

$$\begin{aligned}\operatorname{Re}\frac{\hat{\sigma}}{15}(x+iy\pm z)^6 &= \tilde{\sigma}\operatorname{Re}(x+iy\pm z)^6 - \tilde{\tilde{\sigma}}\operatorname{Im}(x+iy\pm z)^6\\ &> \tilde{\sigma}\operatorname{Re}(x+iy\pm z)^6 - |\tilde{\tilde{\sigma}}|\, B_n^6\end{aligned}$$

Depending on the values of ϕ and z the two Gibbs factors appearing in $f(\phi, z)$, (3.2), are bounded by (4.2) or (4.3). For $B_{n+1} < |\phi| < \sqrt{L}\, B_n$ the pairs "large–large," "small–small," and "large–small" occur, whereas for $|\phi| \geqslant \sqrt{L}\, B_n$ only the pairs "large–large" and "large–small" do.

Lemma. For $|\phi| > B_{n+1}$ with $\phi_1 > 0$, $|\phi_2| < \kappa L^{-1/2} B_{n+1}$, and $0 \leqslant z < \infty$ the function $f(\phi, z)$ satisfies the bound

$$|f(\phi, z)| \leqslant e^{-E}$$

$$E = \tilde{\sigma} L^3 \{ P(x, y) - \tfrac{1}{2} \Omega B_{n+1}^4 z^2 \}$$

$$P(x, y) = \tfrac{1}{2} x^6 - y^2 (15 x^4 + y^4) + D x^4$$

$$\Omega = 15 \left(\frac{64}{L^4} + \frac{15}{2L^2} \right)$$

Proof. (α) The case "large–large." From (4.2) we obtain

$$|f(\phi, z)| \leqslant \exp[-\tilde{\sigma} L^3 \{ P(x, y) + \mathscr{X} + \text{positive terms} \}] \tag{4.4}$$

$$\mathscr{X} = \frac{15}{2} z^2 [x^4 - 12 x^2 y^2] + \frac{15}{2} z^4 [x^2 - 2y^2] + \frac{1}{4} z^6 \tag{4.5}$$

This particular choice of $\mathscr{X}$ will appear in the other cases too.

From the inequality $x^4 - 12x^2y^2 \geqslant -36 y^4$, valid for real x, y, we deduce for $|\phi_2| \leqslant \kappa L^{-1/2} B_{n+1}$

$$x^4 - 12 x^2 y^2 \geqslant -64 \left(\frac{B_{n+1}}{L} \right)^4 \tag{4.6}$$

Moreover

$$x^2 - 2y^2 = \frac{1}{L} [|\phi|^2 - 3\phi_2^2] \geqslant -\frac{B_{n+1}^2}{L} \tag{4.7}$$

From (4.7) together with the inequality

$$-A r^2 + \frac{1}{4} r^3 \geqslant -A^2 r, \qquad \forall A, r \in \mathbb{R}_+ \tag{4.8}$$

we get

$$\frac{15}{2} z^4 [x^2 - 2y^2] + \frac{1}{4} z^6 \geqslant -\frac{15 B_{n+1}^2}{2L} z^4 + \frac{1}{4} z^6$$

$$\geqslant -\left(\frac{15 B_{n+1}^2}{2L} \right)^2 z^2 \tag{4.9}$$

From (4.6) and (4.9) then follows the lower bound for $\mathscr{X}$, (4.5), valid for any real z

$$\mathscr{X} \geqslant -\frac{15}{2} z^2 B_{n+1}^4 \left[\frac{64}{L^4} + \frac{15}{2L^2}\right] \tag{4.10}$$

Hence from (4.4) and (4.10) follows the bound in this case.

(β) The case "large–small." From (4.2) for $+z$ and (4.3) for $-z$ we obtain, discarding some positive terms

$$|f(\phi, z)| \leqslant \exp\left[-\tilde{\sigma}L^3 \left\{P(x, y) - \frac{D}{2}x^4 + \frac{1}{4}(x-z)^6 + \frac{1}{4}z^6 + \mathscr{X} - \frac{1}{2}O(B_n^4) - \frac{1}{2}\frac{|\tilde{\tilde{\sigma}}|}{\tilde{\sigma}} B_n^6\right\}\right] \tag{4.11}$$

Applying

$$(x-z)^6 + z^6 \geqslant 2\left(\frac{x}{2}\right)^6 \tag{4.12}$$

valid for $x, z \in \mathbb{R}$, we can write (4.11) as

$$|f(\phi, z)| \leqslant \exp\left[-\tilde{\sigma}L^3 \left\{P(x, y) + \mathscr{X} - \frac{D}{2}x^4 + \frac{1}{2}\left(\frac{x}{2}\right)^6 - \frac{1}{2}O(B_n^4) - \frac{1}{2}\frac{|\tilde{\tilde{\sigma}}|}{\tilde{\sigma}} B_n^6\right\}\right] \tag{4.13}$$

Since $x \geqslant (3L)^{-1/2} B_{n+1}$ we have for sufficiently large n_0, due to $128(3L)^3 |\tilde{\tilde{\sigma}}| < \tilde{\sigma}$

$$-Dx^4 + \left(\frac{x}{2}\right)^6 - O(B_n^4) - \frac{|\tilde{\tilde{\sigma}}|}{\tilde{\sigma}} B_n^6 > 0 \tag{4.14}$$

reducing this case to the case (α) already treated.

(γ) The case "small–small." The bounds (4.3) lead to, after discarding some positive terms

$$|f(\phi, z)| \leqslant \exp\left[-\tilde{\sigma}L^3 \left\{P(x, y) + \mathscr{X} - Dx^4 + \frac{1}{2}x^6 - O(B_n^4) - \frac{|\tilde{\tilde{\sigma}}|}{\tilde{\sigma}} B_n^6\right\}\right] \tag{4.15}$$

Again, for sufficiently large n_0,

$$-Dx^4 + \frac{1}{2}x^6 - O(B_n^4) - \frac{|\tilde{\tilde{\sigma}}|}{\tilde{\sigma}} B_n^6 > 0 \tag{4.16}$$

reducing this case also to the case (α). Thus the Lemma is proven. ∎

From this lemma follows directly the upper bound

$$\int_{-\infty}^{\infty} d\mu(z)\,|f(\phi, z)| < e^{-\tilde{\sigma}\{(1/2)\phi_1^6 - \phi_2^2(15\phi_1^4 + \phi_2^4) + D\phi_1^4\}} \cdot (1 - \tilde{\sigma}L^3 \Omega B_{n+1}^4)^{-1/2} e^{-\tilde{\sigma}(L-1)D\phi_1^4} \tag{4.17}$$

We assumed again a sufficiently large n_0 such that the argument of the square root is positive. A suitable lower bound for the denominator in (3.1) has already been derived in (3.13). It leads, together with (4.17), to the upper bound for the Gibbs factor (3.1)

$$|g'(\phi)| < e^{-\tilde{\sigma}\{(1/2)\phi_1^6 - \phi_2^2(15\phi_1^4 + \phi_2^4) + D\phi_1^4\}} \cdot \left[\frac{1 + 2\tilde{\sigma}L^3 p B_n^4(1+\varepsilon)}{1 - \tilde{\sigma}L^3 \Omega B_{n+1}^4}\right]^{1/2} e^{-\tilde{\sigma}(L-1)D\phi_1^4} \tag{4.18}$$

In replacing σ by $\hat{\sigma}$ we slightly changed the constant ε. For simplicity we assume, guaranteed by a sufficiently large n_0, $\tilde{\sigma}L^3\Omega B_{n+1}^4 < 1/2$; then

$$\left[\frac{1 + 2\tilde{\sigma}L^3 p B_n^4(1+\varepsilon)}{1 - \tilde{\sigma}L^3 \Omega B_{n+1}^4}\right]^{1/2} \cdot e^{-\tilde{\sigma}(L-1)D\phi_1^4} < e^{\tilde{\sigma}L^3 p B_n^4(1+\varepsilon) + (\sqrt{3}/2)\tilde{\sigma}L^3\Omega B_{n+1}^4 - \tilde{\sigma}(L-1)D\phi_1^4} \tag{4.19}$$

Denoting by $\hat{\sigma}'$ the quantity $\hat{\sigma}$ indexed by $n+1$ we anticipate from (5.39)

$$0 < \operatorname{Re}\hat{\sigma} - \operatorname{Re}\hat{\sigma}' = O(N^{-2}) = \operatorname{Re}\hat{\sigma}O(N^{-1}), \qquad N = n + n_0 \tag{4.20}$$

Recalling the definition (4.1) of $\tilde{\sigma}$, we observe, that in order to replace in the positive term $\tilde{\sigma}\phi_2^2(15\phi_1^4 + \phi_2^4)$ appearing in (4.18) $\operatorname{Re}\hat{\sigma}$ by the weaker $\operatorname{Re}\hat{\sigma}'$, we have to subtract, due to (4.20), a term of the order $\operatorname{Re}\hat{\sigma}B_n^2 N^{-1}\phi_1^4$ which can easily be accounted for by the term proportional to $(L-1)D$. Moreover, from (4.19) and $p(1+\varepsilon) < 2$ we deduce that the choice

$$D = 9L^3\left\{2 + \frac{\sqrt{3}}{2}\Omega\right\}\frac{1}{L-1} \tag{4.21}$$

finally implies in (4.18)

$$|g'(\phi)| < e^{-(1/15)\operatorname{Re}\hat{\sigma}'\{(1/2)\phi_1^6 - \phi_2^2(15\phi_1^4 + \phi_2^4) + D\phi_1^4\}} \tag{4.22}$$

Thus the properties (A_2) are reproduced with the index $n+1$ in place of n.

5. INDUCTIVE REPRODUCTION OF THE SMALL FIELD ASSUMPTIONS

5.1. The Effective Potential

In dealing with the small field region $\phi \in \mathbb{C}$, $|\phi| \leqslant B_{n+1}$ we closely follow Ref. 5, adapting their method to our more involved situation. Let ξ be a fixed positive real number $0 < \xi < 1/4$. On the right hand side of (3.1) the contributions from $|z| \leqslant \xi B_n$ and $|z| > \xi B_n$ are treated separately. Denoting by $\chi(z)$ the characteristic function of the interval $|z| \leqslant \xi B_n$ we write[5]

$$g'(\phi) = g'_1(\phi)[1 + g'_2(\phi)] \tag{5.1}$$

$$g'_1(\phi) = \frac{\int d\mu(z)\, \chi(z)\, f(\phi, z)}{\int d\mu(z)\, \chi(z)\, f(0, z)} \tag{5.2}$$

and

$$g'_2(\phi) = \frac{\int d\mu(z)[1 - \chi(z)]\, f(\phi, z)}{g'_1(\phi) \int d\mu(z)\, f(0, z)} - (\phi = 0) \tag{5.3}$$

We first treat the large z contributions g'_2. Due to $|\phi| \leqslant B_{n+1} < \kappa B_n$ the variables of both Gibbs factors appearing in $f(\phi, z)$ have imaginary parts not greater than $\kappa L^{-1/2} B_n$ in absolute value. In Section 3 we already deduced that there $f(\phi, z)$ is holomorphic in ϕ, ρ_0, λ_0, with z fixed, and satisfies the uniform bound

$$|f(\phi, z)| < M, \qquad \forall z \in \mathbb{R},\ (\rho_0, \lambda_0) \in \mathscr{D}_n \tag{5.4}$$

with M independent of n. Hence

$$\left| \int d\mu(z)[1 - \chi(z)]\, f(\phi, z) \right| < M \int_{|z| > \xi B_n} d\mu(z) < \sqrt{2}\, M\, e^{-(1/2\, \xi B_n)^2} \tag{5.5}$$

Since, as will turn out in the sequel of this section, $g'_1(\phi)$ is holomorphic in $(\phi, \rho_0, \lambda_0)$ for $|\phi| < \kappa B_n$ and bounded from below by a positive constant independent of n, and moreover $\int d\mu(z)\, f(0, z)$ satisfies the lower bound (3.13) we conclude that $g'_2(\phi)$ is holomorphic in $(\phi, \rho_0, \lambda_0)$ for $|\phi| < \kappa B_n$, $(\rho_0, \lambda_0) \in \operatorname{int} \mathscr{D}_n$, satisfying

$$|g'_2(\phi)| < \text{const}\, e^{-(1/2\, \xi B_n)^2} \tag{5.6}$$

Hence, for n_0 large enough, $\ln(1 + g'_2)$ is holomorphic in this domain too, bounded uniformly by

$$|\ln(1 + g'_2(\phi))| < K_2\, e^{-(1/2\, \xi B_n)^2} \tag{5.7}$$

with the constant K_2 not depending on n.

The dominant contributions in the small field region emerge from the small z integral g_1', (5.2), which can be treated by a convergent power series expansion. Choosing the larger domain

$$\phi \in \mathbb{C} \qquad |\phi| \leqslant (1-\xi)\sqrt{L}\, B_n \tag{5.8}$$

one observes, that because of the restricted z integration in (5.2), the variables in the Gibbs factors entering $f(\phi, z)$ are for all ϕ from (5.8) bounded by $|L^{-1/2}\phi \pm z| \leqslant B_n$. Hence the assumptions (A_3) can be used to evaluate g_1'

$$g_1'(\phi) = e^{-(\rho/2)L^2\phi^2 - (\lambda/6)L\phi^4 - (\sigma/15)\phi^6 - (\tau/28L)\phi^8} \cdot h(\phi) \tag{5.9}$$

$$h(\phi) = \frac{\int d\mu(z)\, \chi(z)\, e^{-L^3[v(z) - \tilde{v}(z) + \tilde{V}(\phi,z)] - \zeta}}{\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)}} \tag{5.10}$$

with the definitions

$$\tilde{V}(\phi, z) = \tfrac{1}{2}\{\tilde{v}(L^{-1/2}\phi + z) + \tilde{v}(L^{-1/2}\phi - z)\} \tag{5.11}$$

$$\zeta = \lambda L^2\phi^2 z^2 + \sigma(L\phi^4 z^2 + L^2\phi^2 z^4) + \tau(\phi^6 z^2 + \tfrac{5}{2}L\phi^4 z^4 + L^2\phi^2 z^6) \tag{5.12}$$

We first observe that $h(\phi)$ is holomorphic in $(\phi, \rho_0, \lambda_0)$ for ϕ in the domain (5.8) and $(\rho_0, \lambda_0) \in \mathscr{D}_n$. This is a consequence of the analyticity properties of both integrands due to (A_3) and the finite integration interval, together with the fact that the denominator is $1 + O(N^{-1})$ and thus different from zero. This estimate is deduced from (3.9), replacing B_n there by ξB_n.

It is convenient to introduce the ϕ-dependent complex measure

$$\langle \mathscr{F} \rangle_\phi = \frac{\int d\mu(z)\, \chi(z)\, e^{-L^3[v(z) - \tilde{v}(z) + \tilde{V}(\phi,z)]}\, \mathscr{F}(z)}{\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)}} \tag{5.13}$$

We decompose $h(\phi)$, (5.10), as follows

$$h(\phi) = h_a(\phi) + h_b(\phi) \tag{5.14}$$

$$h_a(\phi) = \langle 1 - \zeta + \tfrac{1}{2}\zeta^2 \rangle_\phi \tag{5.15}$$

$$h_b(\phi) = \langle e^{-\zeta} - 1 + \zeta - \tfrac{1}{2}\zeta^2 \rangle_\phi \tag{5.16}$$

Using for $\zeta \in \mathbb{C}$, $|\zeta| < 3$ the inequality

$$|e^{-\zeta} - 1 + \zeta - \tfrac{1}{2}\zeta^2| < |\zeta|^3 \tag{5.17}$$

to estimate (5.16), possible for n_0 sufficiently large, and the uniform bound for $\tilde{v}$ and hence for $\tilde{V}$ due to (A_3), yields for ϕ in the domain (5.8) and $(\rho_0, \lambda_0) \in \mathscr{D}_n$ the uniform bound

$$\begin{aligned} |h_b(\phi)| &< \frac{e^{2L^3 a N^{-2}} \int d\mu(z)\, \chi(z)\, e^{-L^3 \operatorname{Re} v(z)}\, |\zeta|^3}{|\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)}|} \\ &< K_b B_n^{12} N^{-3} \end{aligned} \tag{5.18}$$

with the constant K_b independent of n. We introduced the shorthand

$$N = n + n_0 \tag{5.19}$$

and estimated the denominator as in (3.7)–(3.10). Finally h_a follows from (5.15) and (5.12) as

$$\begin{aligned} h_a(\phi) = {} & \langle 1 \rangle_\phi - L^2\phi^2\{\lambda\langle z^2\rangle_\phi + \sigma\langle z^4\rangle_\phi + \tau\langle z^6\rangle_\phi\} \\ & - L\phi^4 \left\{\sigma\langle z^2\rangle_\phi - \frac{1}{2} L^3\lambda^2\langle z^4\rangle_\phi + \frac{5}{2}\tau\langle z^4\rangle_\phi \right. \\ & \left. - \frac{1}{2} L^3\sigma^2\langle z^8\rangle_\phi - L^3\lambda\sigma\langle z^6\rangle_\phi + O(N^{-3})\right\} \\ & + L^3\phi^6\left\{-\frac{\tau}{L^3}\langle z^2\rangle_\phi + \lambda\sigma\langle z^4\rangle_\phi + \sigma^2\langle z^6\rangle_\phi + O(N^{-3})\right\} \\ & + \frac{1}{2} L^2\phi^8\{\sigma^2\langle z^4\rangle_\phi + O(N^{-3})\} \\ & + \phi^{10} O(N^{-3}) + \phi^{12} O(N^{-4}) \end{aligned} \tag{5.20}$$

The order estimates are valid uniformly in ϕ, ρ_0, λ_0. This follows from the estimate of the "expectations" $\langle z^{2k}\rangle_\phi$, (5.13); using the estimate $1 + O(N^{-1})$ for the denominator and writing

$$\begin{aligned} & \int d\mu(z)\, \chi(z)\, z^{2k}\, e^{-L^3[v(z) - \tilde{v}(z) + \tilde{V}(\phi, z)]} \\ & \quad = \int_{-\infty}^{\infty} d\mu(z)\, z^{2k} - \int_{|z| > \xi B_n} d\mu(z)\, z^{2k} \\ & \qquad + \int_{|z| < \xi B_n} d\mu(z)\, z^{2k}\{e^{-L^3[v(v) - \tilde{v}(z) + \tilde{V}(\phi, z)]} - 1\} \end{aligned} \tag{5.21}$$

we deduce from (A_3), uniformly in ϕ, (5.8), and ρ_0, λ_0

$$\langle z^{2k}\rangle_\phi = (2k-1)!! + O(N^{-1}) + O(N^{-2}) \tag{5.22}$$

where $O(N^{-1})$ does not depend on ϕ. Moreover we write

$$\langle 1 \rangle_\phi = 1 + \Delta(\phi) \tag{5.23}$$

$$\Delta(\phi) = \frac{\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)} \{ e^{L^3[\tilde{v}(z) - \tilde{V}(\phi, z)]} - 1 \}}{\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)}} \tag{5.24}$$

In order to obtain an optimal bound we decompose

$$\Delta(\phi) = \Delta_1 + \Delta_2 \tag{5.25}$$

$$\Delta_1 = \frac{\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)} \{ e^{L^3 \tilde{v}(z)} - 1 \}}{\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)}} \tag{5.26}$$

$$\Delta_2 = \frac{\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)} \{ e^{-L^3 \tilde{V}(\phi, z)} - 1 \}\, e^{L^3 \tilde{v}(z)}}{\int d\mu(z)\, \chi(z)\, e^{-L^3 v(z)}} \tag{5.27}$$

In order to estimate (5.26) we first observe that

$$\tilde{v}(z) = z^{10} w(z) \tag{5.28}$$

with $w(z)$ holomorphic in $|z| < B_n$. From the maximum modulus theorem we obtain

$$|\tilde{v}(z)| < \left(\frac{|z|}{B_n} \right)^{10} a N^{-2} \tag{5.29}$$

which we use to bound (5.26). In (5.27) we can use in the domain considered $|\tilde{V}(\phi, z)| < aN^{-2}$ and thus obtain for n_0 large enough the uniform bound

$$|\Delta(\phi)| < \frac{aL^3}{N^2} \left\{ 1 + \frac{c}{B_n^{10}} \right\} (1 + \varepsilon_\Delta) \tag{5.30}$$

where ε_Δ is a small positive number of order N^{-2} and $c > 0$ a constant.

From the bounds (5.18), (5.20), (5.22), (5.30) we conclude that for n_0 large enough the function $h(\phi)$ is holomorphic in $(\phi, \rho_0, \lambda_0)$ with ϕ in the domain (5.8) and $(\rho_0, \lambda_0) \in \mathscr{D}_n$, and satisfies $|h(\phi) - 1| < 1$ there. Hence $\ln h(\phi)$ is holomorphic in the same domain. Choosing $\alpha < 1/12$ in the assumption (A_3) we get

$$\ln h(\phi) = h_a(\phi) - 1 - \frac{1}{2} (h_a(\phi) - 1)^2 + O(N^{-3+12\alpha}) \tag{5.31}$$

The holomorphic function $\ln h(\phi)$ has the power series representation in ϕ

$$-\ln h(\phi) = \frac{\rho_1}{2} \phi^2 + \frac{\lambda_1}{6} \phi^4 + \frac{\sigma_1}{15} \phi^6 + \frac{\tau_1}{28} \phi^8 + \tilde{v}_1(\phi) \tag{5.32}$$

where $\tilde{v}_1$ is the sum of the even powers in ϕ higher than eight. Moreover ρ_1, λ_1, σ_1, τ_1 as well as $\tilde{v}_1$ are holomorphic functions of $(\rho_0, \lambda_0) \in \mathscr{D}_n$. Using the Cauchy formula for derivatives we obtain from (5.31) and (5.20)

$$\begin{aligned}
\frac{\rho_1}{2} &= L^2\{\lambda + 3\sigma + O(N^{-2})\} \\
\frac{\lambda_1}{6} &= L\{\sigma + O(N^{-2})\} \\
\frac{\sigma_1}{15} &= L^3\left\{-2\lambda\sigma - 12\sigma^2 + \frac{\tau}{L^3} + O(N^{-2-6\alpha})\right\} \\
\frac{\tau_1}{28} &= -L^2\sigma^2 + O(N^{-2-8\alpha})
\end{aligned} \tag{5.33}$$

Furthermore (5.30) and the Cauchy formula for derivatives imply for the power series $\tilde{v}_1$ of (5.32) a uniform bound in the restricted domain $|\phi| \leqslant B_{n+1}$, $(\rho_0, \lambda_0) \in \mathscr{D}_n$

$$|\tilde{v}_1(\phi)| \leqslant \frac{1+\varepsilon'_\Delta}{(1-\xi)^{10} L^2} \frac{(B_{n+1}/B_n)^{10}}{1 - [B_{n+1}/(1-\xi)\sqrt{L}\, B_n]^2} \frac{a}{N^2} \tag{5.34}$$

where $\varepsilon'_\Delta > 0$ is a small constant, and n_0 large enough. We observe that this bound is smaller than $a(N+1)^{-2}$ even with $L = 2$ if ξ is chosen sufficiently small and n_0 sufficiently large.

Collecting pieces, we finally infer from (5.1), (5.7), (5.9), and (5.32) that $\ln g'(\phi)$ is holomorphic in $(\phi, \rho_0, \lambda_0)$ in the domain $|\phi| \leqslant B_{n+1}$, $(\rho_0, \lambda_0) \in \operatorname{int} \mathscr{D}_n$

$$v'(\phi) \equiv -\ln g'(\phi) = \frac{\rho'}{2}\phi^2 + \frac{\lambda'}{6}\phi^4 + \frac{\sigma'}{15}\phi^6 + \frac{\tau'}{28}\phi^8 + \tilde{v}'(\phi) \tag{5.35}$$

with $\tilde{v}'$ being the sum of the even powers higher than eight. In addition, with (5.33),

$$\begin{aligned}
\rho' &= L^2\rho + \rho_1 \\
\lambda' &= L\lambda + \lambda_1 \\
\sigma' &= \sigma + \sigma_1 \\
\tau' &= L^{-1}\tau + \tau_1 \\
\tilde{v}'(\phi) &= \tilde{v}_1(\phi) + O(e^{-(1/2\,\xi B_n)^2})
\end{aligned} \tag{5.36}$$

In writing the equations for $\rho', \ldots, \tau'$, which are holomorphic in $(\rho_0, \lambda_0) \in \operatorname{int} \mathscr{D}_n$, we suppressed the exponentially small orders. What remains to be proven in reproducing the assumptions (A_3) with index $n+1$ are the bounds on the coupling constants ρ', λ', σ', τ'.

5.2. The Flow of the Coupling Constants

The recursion relations (5.36) determine the flow of the couplings ρ_n, λ_n, σ_n, τ_n up to corrections. Assuming convergence to zero we obtain from (5.36) the asymptotic behavior

$$\begin{aligned} \sigma_n &\sim \frac{L-1}{240\, L^3} \cdot \frac{1}{N} \\ \rho_n &\sim 6 \frac{L^2}{(L-1)^2} \sigma_n \\ \lambda_n &\sim -6 \frac{L}{L-1} \sigma_n \\ \tau_n &\sim -28 \frac{L^3}{L-1} \sigma_n^2 \end{aligned} \tag{5.37}$$

for $n \to \infty$, $N = n + n_0$, n_0 large. This is in accordance with the bounds assumed in (A_3).

Due to the higher order corrections in (5.36) we have to control the iteration of the couplings using intervals or complex neighborhoods around the asymptotic values (5.37).

This proves to be feasible only after decoupling at least the equation of the marginal variable from the other ones to second order, using a nonlinear transformation to a variable $\hat{\sigma}$. We simplify (5.36) further by "diagonalizing" the other recursion relations to leading order. Thus we define

$$\begin{aligned} \hat{\rho}_n &= \rho_n + 2 \frac{L}{L-1} \lambda_n + 6 \left(\frac{L}{L-1} \right)^2 \sigma_n \\ \hat{\lambda}_n &= \lambda_n + 6 \frac{L}{L-1} \sigma_n \\ \hat{\sigma}_n &= \sigma_n + 15 \frac{L}{L-1} \hat{\tau}_n + 30 \frac{L^3}{L-1} \hat{\lambda}_n \sigma_n \\ \hat{\tau}_n &= \tau_n + 28 \frac{L^3}{L-1} \sigma_n^2 \end{aligned} \tag{5.38}$$

and infer from (5.36) and the bounds on ρ_n, λ_n, σ_n, τ_n in (A_3)

$$\begin{aligned} \hat{\rho}_{n+1} &= L^2\hat{\rho}_n + O(N^{-2}) \\ \hat{\lambda}_{n+1} &= L\hat{\lambda}_n + O(N^{-2}) \\ \hat{\sigma}_{n+1} &= \hat{\sigma}_n - \frac{240\, L^3}{L-1}\hat{\sigma}_n^2 + O(N^{-2-6\alpha}) \\ \hat{\tau}_{n+1} &= \frac{1}{L}\hat{\tau}_n + O(N^{-2-8\alpha}) \end{aligned} \tag{5.39}$$

The coefficients $\hat{\rho}_n$, $\hat{\lambda}_n$ are connected with normal ordering. Defining

$$:F(\phi): = e^{-(1/2)(L/L-1)(d/d\phi)^2}\, F(\phi) \tag{5.40}$$

we observe that the effective potential has the form

$$v_n(\phi) = \frac{\hat{\rho}_n}{2} :\phi^2: + \frac{\hat{\lambda}_n}{6} :\phi^4: + \frac{\sigma_n}{15} :\phi^6: + \frac{\tau_n}{28}\phi^8 + \tilde{v}_n(\phi) + \text{const} \tag{5.41}$$

We analyze the recursion relations (5.39), considering initial values $\tau_0 = 0$ or of order n_0^{-2}, $\sigma_0 \in \mathbb{R}_+$, $(\hat{\rho}_0, \hat{\lambda}_0) \in \hat{\mathscr{D}}_0 \subset \mathbb{C}^2$, with σ_0, $|\hat{\rho}_0|$, $|\hat{\lambda}_0|$ of order n_0^{-1}. $\hat{\mathscr{D}}_0$ is specified below. It is almost trivial to see that these initial Gibbs factors satisfy our inductive assumptions if n_0 is sufficiently large.

From (5.39) we conclude that for fixed $R_\tau \in \mathbb{R}_+$, chosen large enough to include $\tau_0 = 0$, and n_0 large enough

$$|\hat{\tau}_n| < R_\tau N^{-2} \Rightarrow |\hat{\tau}_{n+1}| < R_\tau (N+1)^{-2} \tag{5.42}$$

In order to treat the marginally stable σ_n we define for $n \in \mathbb{N}$

$$\hat{\sigma}_n = \frac{L-1}{240\, L^3} \cdot \frac{1+s_n}{N}, \qquad s_n \in \mathbb{C} \tag{5.43}$$

and assume, for fixed $R_\sigma \in \mathbb{R}_+$

$$|s_n| < R_\sigma \tag{5.44}$$

We choose

$$R_\sigma < \frac{1}{128(3L)^3 + 1} \tag{5.45}$$

in order to satisfy $128(3L)^3\, |\text{Im}\, \hat{\sigma}| < \text{Re}\, \hat{\sigma}$ required in (A_3). From (5.39) we obtain for $\hat{\sigma}_{n+1}$

$$s_{n+1} = s_n \left[1 - \frac{1+s_n}{N} + O(N^{-1-6\alpha}) \right] \tag{5.46}$$

and thus conclude, for n_0 sufficiently large

$$|s_n| < R_\sigma \Rightarrow |s_{n+1}| < R_\sigma \tag{5.47}$$

We are left with the task to find critical values $\rho_0 = \rho^*$, $\lambda_0 = \lambda^*$ for the relevant couplings of the initial Gibbs factor (1.3), such that $|\rho_n|$, $|\lambda_n|$ stay bounded of order $N^{-1} = (n + n_0)^{-1}$ for all n. For this purpose we exploit analyticity of $g^{(n)}(\phi)$ in $(\rho_0, \lambda_0) \in \mathscr{D}_n \subset \mathbb{C}^2$, with successively restricted compact domains $\mathscr{D}_n \subset \operatorname{int} \mathscr{D}_{n-1}$. In Section 3 we have already proven that holomorphy of $g^{(n)}(\phi)$ in $(\lambda_0, \rho_0) \in \mathscr{D}_n$ implies holomorphy of $g^{(n+1)}(\phi)$ in $(\rho_0, \lambda_0) \in \operatorname{int} \mathscr{D}_n$. Our analysis is simplified introducing $(\hat\rho_n, \hat\lambda_n)$, (5.38). Due to this linear transformation the Gibbs factors obviously inherit for $(\hat\rho_0, \hat\lambda_0)$ holomorphy properties from those in (ρ_0, λ_0) with domains $\hat{\mathscr{D}}_n$ and $\mathscr{D}_n$, respectively. Thus from (5.35) we deduce, that we have holomorphic mappings, $n \in \mathbb{N}$

$$\begin{aligned} H_n\colon\quad &\operatorname{int} \hat{\mathscr{D}}_{n-1} \to \mathbb{C}^2 \\ &(\hat\rho_0. \hat\lambda_0) \mapsto (\hat\rho_n, \hat\lambda_n) \end{aligned} \tag{5.48}$$

In order to specify the domains $\hat{\mathscr{D}}_n$, we introduce the families of polydisks, with a constant $R > 0$

$$\mathscr{P}_n = \{(\hat\rho_n, \hat\lambda_n) \in \mathbb{C}^2 : |\hat\rho_n| < RN^{-1}, |\hat\lambda_n| < RN^{-1}\} \tag{5.49}$$

$$\mathscr{Q}_n = \{(\hat\rho_n, \hat\lambda_n) \in \mathbb{C}^2 : |\hat\rho_n| < L^{-1/2}RN^{-1}, |\hat\lambda_n| < L^{-1/2}RN^{-1}\} \tag{5.50}$$

the norm, $(z_1, z_2) \in \mathbb{C}^2$

$$\|(z_1, z_2)\|_1 = |z_1| + |z_2| \tag{5.51}$$

and define the diameter of a domain $\mathscr{B} \subset \mathbb{C}^2$ as

$$\operatorname{diam} \mathscr{B} = \sup_{\zeta, \zeta' \in \mathscr{B}} \|\zeta - \zeta'\|_1 \tag{5.52}$$

We then choose

$$\hat{\mathscr{D}}_0 = \bar{\mathscr{P}}_0 \tag{5.53}$$

i.e., the closure of $\mathscr{P}_0$, and prove inductively.

(A$'_3$) The holomorphic maps H_n, (5.48), $n \in \mathbb{N}$, have bijective restrictions

$$H_n|_{\hat{\mathscr{D}}_n} : \hat{\mathscr{D}}_n \to \bar{\mathscr{P}}_n$$

implying

$$\operatorname{diam} \hat{\mathscr{D}}_n \leqslant 4R(L - 2\varepsilon)^{-n} N^{-1}$$

with $(L - 2\varepsilon) > 1$ independent of n.

Proof. We prove (A_3') for $n+1$ assuming it to be true for n. We observe[9] that H_n^{-1} is holomorphic too and define the holomorphic map

$$h_{n+1} = H_{n+1} \circ H_n^{-1}: \qquad \mathscr{P}_n \to \mathbb{C}^2 \qquad\qquad (\hat{\rho}, \hat{\lambda}_n) \mapsto (\hat{\rho}_{n+1}, \hat{\lambda}_{n+1}) \tag{5.54}$$

where for $n = 0$ we have to put $H_0 = \mathrm{id}$.

From (5.39) we obtain for h_{n+1} the system

$$\begin{aligned} \hat{\rho}_{n+1} &= L^2 \hat{\rho}_n + F_n(\hat{\rho}_n, \hat{\lambda}_n) \\ \hat{\lambda}_{n+1} &= L\hat{\lambda}_n + G_n(\hat{\rho}_n, \hat{\lambda}_n) \end{aligned} \tag{5.55}$$

with holomorphic functions F_n, G_n bounded on the domain considered by $O(N^{-2})$. Let $(\hat{\rho}_n, \hat{\lambda}_n)$, $(\hat{\rho}'_n, \hat{\lambda}'_n) \in \mathscr{Q}_n$, (5.50). Using the Cauchy integral representation for F_n and G_n with integrations performed on the distinguished boundary $\mathscr{C}_n$ of a polydisk $\mathscr{P}'_n$, with radii R' smaller but close to R, i.e.

$$\mathscr{C}_n = \{(z, w) \in \mathbb{C}^2 \colon |z| = |w| = R'N^{-1}\} \tag{5.56}$$

yields

$$\begin{aligned} &F_n(\hat{\rho}_n, \hat{\lambda}_n) - F_n(\hat{\rho}'_n, \hat{\lambda}'_n) \\ &\quad = (2\pi i)^{-2} \int_{\mathscr{C}_n} dz \int dw F_n(z, w) \left\{ \frac{1}{(z - \hat{\rho}_n)(w - \hat{\lambda}_n)} - \frac{1}{(z - \hat{\rho}'_n)(w - \hat{\lambda}'_n)} \right\} \\ &\quad = (2\pi i)^{-2} \int_{\mathscr{C}_n} dz \int dw F_n(z, w) \left\{ \frac{\hat{\rho}_n - \hat{\rho}'_n}{(w - \hat{\lambda}_n)(z - \hat{\rho}_n)(z - \hat{\rho}'_n)} \right. \\ &\qquad \left. + \frac{\hat{\lambda}_n - \hat{\lambda}'_n}{(z - \hat{\rho}'_n)(w - \hat{\lambda}_n)(w - \hat{\lambda}'_n)} \right\} \end{aligned} \tag{5.57}$$

and a similar equation in the case of G_n. Since all the differences occurring in the denominators of (5.57) have the common lower bound $(1 - L^{-1/2})\, RN^{-1}$ and because F_n and G_n are of the order N^{-2} we easily deduce

$$|F_n(\hat{\rho}_n, \hat{\lambda}_n) - F_n(\hat{\rho}'_n, \hat{\lambda}'_n)| \leqslant \varepsilon_n \, \|(\hat{\rho}_n - \hat{\rho}'_n, \hat{\lambda}_n - \hat{\lambda}'_n)\|_1 \tag{5.58}$$

with $\varepsilon_n = O(N^{-1}) < \varepsilon$, $\varepsilon > 0$ a small constant, and the identical bound for the case of G_n. From (5.55) and (5.58) we finally infer for $(\hat{\rho}_n, \hat{\lambda}_n)$, $(\hat{\rho}'_n, \hat{\lambda}'_n) \in \mathscr{Q}_n$

$$\|(\hat{\rho}_{n+1} - \hat{\rho}'_{n+1}, \hat{\lambda}_{n+1} - \hat{\lambda}'_{n+1})\|_1 \geqslant (L - 2\varepsilon) \, \|(\hat{\rho}_n - \hat{\rho}'_n, \hat{\lambda}_n - \hat{\lambda}'_n)\|_1 \tag{5.59}$$

Hence $h_{n+1}|_{\mathscr{Q}_n}$ is injective. Moreover, (5.59) shows that $h_{n+1}|_{\mathscr{Q}_n}$ is expansive with an expansion parameter bounded below by $(L - 2\varepsilon) > 1$. From (5.55) and the bounds on F_n, G_n it is obvious that $h_{n+1}(\mathscr{Q}_n) \supset \bar{\mathscr{P}}_{n+1}$, for n_0 large. This implies that the restriction of H_{n+1} to $\hat{\mathscr{D}}_{n+1} \subset \text{int}\, \hat{\mathscr{D}}_n$ exists, with

$$H_{n+1}|_{\hat{\mathscr{D}}_{n+1}}: \hat{\mathscr{D}}_{n+1} \to \bar{\mathscr{P}}_{n+1} \tag{5.60}$$

bijective and holomorphic. The lower bound on the expansion factor implies the claim for the diameter. ▮

From (A_3') we infer the existence of a unique critical point

$$\{(\hat{\rho}_0, \hat{\lambda}_0)_{\text{crit}}\} = \{(\rho^*, \lambda^*)\} = \bigcap_{n=0}^{\infty} \hat{\mathscr{D}}_n \tag{5.61}$$

which is a real pair, since the mappings H_n are real analytic for all n. It is now obvious that because of (5.38), (5.42), (5.47), and (5.60) we proved inductively bounds on ρ_n, λ_n, σ_n, τ_n as stated in (A_3), valid under the successive restrictions $\mathscr{D}_0 \supset \mathscr{D}_1 \supset \mathscr{D}_2 \cdots$ on (ρ_0, λ_0) and thus valid for the critical point (5.61) for all n.

For the unique real-analytic critical Gibbs factor $g^{(0)}(\phi)$ found by (5.61) within a family (1.3) for σ_0 small enough, we thus proved convergence of the iterates $g^{(n)}(\phi)$, (1.1), uniformly on compacts $K \subset \mathbb{C}$

$$g^{(n)}(\phi) \xrightarrow[n \to \infty]{} 1 \tag{5.62}$$

where the domain of holomorphy of $g^{(n)}(\phi)$ includes K eventually for all n.

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